

Jacobi Quasi-Nijenhuis Algebroids *

Yunhe Sheng

School of Mathematics, Jilin University, Changchun 130012, Jilin, China

and

School of Mathematics, Dalian University of Technology, Dalian 116024, China

email: ysheng888@gmail.com

Abstract

In this paper, for a Jacobi algebroid (A, ρ) , by introducing the notion of Jacobi quasi-Nijenhuis algebroids, which is a generalization of Poisson quasi-Nijenhuis manifolds introduced in [31] by Stiénon and Xu, we study generalized complex structures on the Courant-Jacobi algebroid $A \oplus A^*$, which unify generalized complex (contact) structures on an even(odd)-dimensional manifold.

1 Introduction

The notion of Poisson quasi-Nijenhuis manifolds was introduced in [31] by Stiénon and Xu. In [1], the author studied Poisson quasi-Nijenhuis structures with background. One can study generalized complex structures in term of Poisson quasi-Nijenhuis structures. Generalized complex structures were introduced by Hitchin [15] and further studied by Gualtieri [14] as a bridge of symplectic and complex structures. Note that only on even-dimensional manifolds, there are generalized complex structures. In [19], Iglesias-Ponte and Wade gave the odd-dimensional analogue of the concept of generalized complex structures under the name of generalized contact structures.

Jacobi structures on a manifold M are local Lie algebra structures [20] on $C^\infty(M)$. It contains a bi-vector field Λ and a vector field X such that $[\Lambda, \Lambda] = 2X \wedge \Lambda$ and $[X, \Lambda] = 0$. In [18], Iglesias and Marrero introduced the notion of generalized Lie bialgebroids in such a way that the base manifold is a Jacobi manifold. The same object was introduced in [12] by Grabowski and Marmo under the name of Jacobi bialgebroids. Similar as the fact that the double of a Lie bialgebroid is a Courant algebroid, the double of a generalized Lie bialgebroid (Jacobi bialgebroid) is a generalized Courant algebroid (Courant-Jacobi algebroid). These topics are widely studied [2], [3], [4], [11], [12], [13], [18], [26], [27], [28].

In this paper, for a Jacobi algebroid (A, ρ) , we study Jacobi quasi-Nijenhuis structures. As an application, we study generalized complex structures on the Courant-Jacobi algebroid $A \oplus A^*$, which unify generalized complex structures on an even-dimensional manifold and generalized contact structures on an odd-dimensional manifold. By definition, a Jacobi quasi-Nijenhuis algebroid is a quadruple $((A, \rho), \pi, N, \phi)$, where (A, ρ) is a Jacobi algebroid, $\pi \in \Gamma(\wedge^2 A)$ is a Jacobi bi-vector field, $N \in \Gamma(A^* \otimes A)$ is compatible with π , and $\phi \in \Gamma(\wedge^3 A^*)$ satisfying $\mathfrak{d}\phi = 0$ and $\mathfrak{d}(i_N \phi) = 0$, such that the Nijenhuis torsion $T(N)$ of N can be expressed as

$$T(N)(X, Y) = \pi^\sharp(i_{X \wedge Y} \phi), \quad \forall X, Y \in \Gamma(A).$$

We generalize some well known results and formulas which hold in the case of Poisson quasi-Nijenhuis manifolds. The biggest obstruction is that in the frame work of “Jacobi” world, the differential and the

⁰*Keyword:* quasi-Jacobi bialgebroids, Jacobi quasi-Nijenhuis algebroids, generalized complex structures, generalized contact structures, locally conformal symplectic structures

⁰*MSC:* Primary 17B65. Secondary 18B40, 58H05.

*Research partially supported by NSF of China (10871007), China Postdoctoral Science Foundation, Science Research Foundation for Excellent Young Teachers of College of Mathematics at Jilin University, US-China CMR Noncommutative Geometry (10911120391/A0109).

Lie derivative are no longer derivations with respect to the wedge product, \wedge . A Generalized complex structure is defined in the usual way, it is a bundle map $\mathcal{J} : A \oplus A^* \longrightarrow A \oplus A^*$ preserving the canonical pairing and satisfying $\mathcal{J}^2 = -\text{Id}$ as well as the integrability condition, which is expressed in term of the Courant-Jacobi bracket. Since the usual Courant algebroid and $\mathcal{E}^1(M)$ are special Courant-Jacobi algebroids, thus it unifies generalized complex structures on an even-dimensional manifold and generalized contact structures on an odd-dimensional manifold.

The paper is organized as follows. In Section 2 we proved that there is a one-to-one correspondence between quasi-Jacobi bialgebroids and quasi-Manin triples. In Section 3 we introduce the notion of Jacobi quasi-Nijenhuis algebroids and give the relation with quasi-Jacobi bialgebroids. In Section 4 we study generalized complex structures on the Courant-Jacobi algebroid $A \oplus A^*$. We prove that there is also a one-to-one correspondence between generalized complex structures and Jacobi quasi-Nijenhuis algebroids satisfying a homomorphism condition. In Section 5 we study generalized complex structures on $TM \oplus T^*M$ for an even-dimensional manifold M and we will see that how a conformal symplectic structure is involved in a generalized complex structure. In Section 6 we study generalized complex structures on $\mathcal{E}^1(M)$ for an odd-dimensional manifold M . Since $(TM \oplus \mathbb{R}, \text{Id})$ is a natural Jacobi algebroid, we recover the notion of generalized contact structures introduced in [19]. Some examples are also discussed.

Notations: We denote the usual Lie bracket of vector fields or the Lie bracket on a Lie algebroid by $[\cdot, \cdot]$, the bracket of the Schouten-Jacobi algebra decided by a Jacobi algebroid by $\llbracket \cdot, \cdot \rrbracket$, the bracket on a Courant-Jacobi algebroid by $[\cdot, \cdot]$. d is the usual deRham differential or the differential associated with a Lie algebroid. \mathfrak{d} is the differential associated with a Jacobi algebroid. For any $X \in \Gamma(A)$, where (A, ρ) is a Jacobi algebroid, L_X is the usual Lie derivative decided by the Lie algebroid structure and \mathfrak{L}_X is the Lie derivative decided by the Jacobi algebroid structure. $\mathbf{1}$ is the constant function with the value 1. Id is the identity map if there is no special explanation.

Acknowledgements: We would like to give our special thanks to Zhangju Liu for very helpful comments and also give our warmest thanks to Chenchang Zhu for the help during we stayed in Courant Research Center, Göttingen, where a part of work was done. We also give our warmest thanks to the referees for many helpful suggestions and pointing out typos and erroneous statements.

2 Quasi-Manin triples

A Lie algebroid over a manifold M is a vector bundle $A \longrightarrow M$ together with a Lie bracket $[\cdot, \cdot]$ on the section space $\Gamma(A)$ and a bundle map $a : A \longrightarrow TM$, called the anchor, satisfying the compatible condition:

$$[X, fY] = f[X, Y] + a(X)(f)Y, \quad \forall X, Y \in \Gamma(A), f \in C^\infty(M).$$

We usually denote a Lie algebroid by $(A, [\cdot, \cdot], a)$, or A if there is no confusion. For a $(1, 1)$ -tensor $N \in \Gamma(A^* \otimes A)$, the Nijenhuis torsion $T(N) : \wedge^2 A \longrightarrow A$ is defined by

$$T(N)(X, Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]), \quad \forall X, Y \in \Gamma(A). \quad (1)$$

If $T(N) = 0$, N is called a Nijenhuis operator on the Lie algebroid A . We can also introduce a new bracket $[\cdot, \cdot]_N$ on $\Gamma(A)$ which is defined as follows:

$$[X, Y]_N = [NX, Y] + [X, NY] - N[X, Y], \quad \forall X, Y \in \Gamma(A). \quad (2)$$

If N is a Nijenhuis operator, $[\cdot, \cdot]_N$ is also a Lie bracket and N is a Lie algebroid morphism from Lie algebroid $(A, [\cdot, \cdot]_N, a \circ N)$ to Lie algebroid $(A, [\cdot, \cdot], a)$.

For any $\pi \in \Gamma(\wedge^2 A)$ and $\sigma \in \Gamma(\wedge^2 A^*)$, $\pi^\sharp : A^* \longrightarrow A$ and $\sigma_\flat : A \longrightarrow A^*$ are given by

$$\pi^\sharp(\xi)(\eta) = \pi(\xi, \eta), \quad \sigma_\flat(X)(Y) = \sigma(X, Y), \quad \forall \xi, \eta \in \Gamma(A^*), \forall X, Y \in \Gamma(A).$$

For any $N \in \Gamma(A^* \otimes A)$ and $\pi \in \Gamma(\wedge^2 A)$, $\pi_N \in \Gamma(\wedge^2 A)$ is defined by

$$\pi_N(\xi, \eta) = \eta(N\pi^\sharp(\xi)), \quad \forall \xi, \eta \in \Gamma(A^*).$$

A Jacobi algebroid is a Lie algebroid $(A, [\cdot, \cdot], a)$ together with a 1-cocycle $\phi_0 \in \Gamma(A^*)$ and we denote it by (A, ϕ_0) . There is a ϕ_0 -bracket $[\cdot, \cdot]_{\phi_0}$ on $\Gamma(\wedge^\bullet A)$, which is given by

$$[P, Q]_{\phi_0} = [P, Q] + (-1)^{p+1}(p-1)P \wedge i_{\phi_0}Q - (q-1)i_{\phi_0}P \wedge Q, \quad (3)$$

for any $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^q A)$. The ϕ_0 -differential d^{ϕ_0} and the ϕ_0 -Lie derivative $L_X^{\phi_0}$ [18] are defined by

$$d^{\phi_0}c = dc + \phi_0 \wedge c, \quad L_X^{\phi_0} = i_X d^{\phi_0} + d^{\phi_0} i_X.$$

In fact, a Jacobi algebroid (A, ϕ_0) is equivalent to the Lie algebroid $(A, [\cdot, \cdot], a)$ together with a representation $\rho : A \rightarrow \mathfrak{D}(M \times \mathbb{R})$ on the trivial line bundle $M \times \mathbb{R}$, where $\mathfrak{D}(M \times \mathbb{R})$ is the gauge Lie algebroid of $M \times \mathbb{R}$. The representation is given by

$$\rho(u)(f) = a(u)f + \phi_0(u)f, \quad \forall u \in \Gamma(A), f \in C^\infty(M) = \Gamma(M \times \mathbb{R}). \quad (4)$$

One can easily prove that ρ is a representation if and only if ϕ_0 is a 1-cocycle. More generally we have

Lemma 2.1. *For any $\theta \in \Gamma(A^* \otimes (M \times \mathfrak{gl}(n)))$, i.e. $\mathfrak{gl}(n)$ -valued 1-form on A , $\rho = a + \theta$ is a representation on $M \times \mathbb{R}^n$ if and only if θ satisfies the Maurer-Cartan equation, more precisely,*

$$d\theta + \frac{1}{2}[\theta \wedge \theta] = 0.$$

Proof. By straightforward computations, we have

$$\begin{aligned} [\rho(X), \rho(Y)] &= [a(X) + \theta(X), a(Y) + \theta(Y)] \\ &= [a(X), a(Y)] + [\theta(X), \theta(Y)] + a(X)\theta(Y) - a(Y)\theta(X) \end{aligned}$$

On the other hand, $\rho([X, Y]) = a([X, Y]) + \theta([X, Y])$, therefore, after comparing the values in TM and $M \times \mathfrak{gl}(n)$, we obtain the required result. ■

Conversely, for the Lie algebroid $(A, [\cdot, \cdot], a)$ and a representation $\rho : A \rightarrow \mathfrak{D}(M \times \mathbb{R})$, denote by $d : \Gamma(\wedge^\bullet A^*) \rightarrow \Gamma(\wedge^{\bullet+1} A^*)$ the associated differential operator, i.e.

$$\begin{aligned} dc(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i \rho(X_i) c(X_0, \dots, \widehat{X_i}, \dots, X_k) \\ &\quad + \sum_{i < j} (-1)^{i+j} c([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k). \end{aligned} \quad (5)$$

Then we can obtain a 1-cocycle $d\mathbf{1} \in \Gamma(A^*)$. Obviously, if the representation ρ is given by (4), then

$$\phi_0 = d\mathbf{1}, \quad d\omega = d\omega + \phi_0 \wedge \omega, \quad \forall \omega \in \Gamma(\wedge^k A^*).$$

Therefore, we have $d = d^{\phi_0}$, the ϕ_0 -differential. Consequently for any $X \in \Gamma(A)$, we can define the Lie derivative $\mathfrak{L}_X : \Gamma(\wedge^k A^*) \rightarrow \Gamma(\wedge^k A^*)$ by Cartan formula:

$$\mathfrak{L}_X = i_X \circ d + d \circ i_X.$$

Obviously, we have $\mathfrak{L}_X \omega = L_X \omega + \phi_0(X)\omega$, which implies $\mathfrak{L}_X = L_X^{\phi_0}$, the ϕ_0 -Lie derivative.

Remark 2.2. *We should be very careful that since d is no longer a derivation, \mathfrak{L}_X is not a derivation. Therefore, the induced Lie derivative $\mathfrak{L}_X : \Gamma(\wedge^k A) \rightarrow \Gamma(\wedge^k A)$ is also not a derivation. This Lie derivative is exactly the foundation of the ϕ_0 -bracket introduced in [18]. Certainly, by this Lie derivative we can only define the ϕ_0 -bracket of a 1-vector field and a k -vector field, and then by some rules one can obtain the bracket of any l -vector field and any k -vector field, see also [12] and [13] for more details.*

Convention: We denote a Jacobi algebroid by (A, ρ) and the associated Schouten-Jacobi algebra by $(\Gamma(\wedge^\bullet A), \llbracket \cdot, \cdot \rrbracket)$.

The notion of Courant-Jacobi algebroids was introduced in [13]. In [18], the authors proved that they are the same as generalized Courant algebroids. They are generalizations of Courant algebroids introduced in [23], see also [29]. In fact, Courant algebroids and Courant-Jacobi algebroids are all special cases of E -Courant algebroids introduced in [8], where E is a vector bundle.

Definition 2.3. A Courant-Jacobi algebroid is a vector bundle \mathcal{K} over M together with

- (1) a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle;
- (2) a bilinear operator \circ on $\Gamma(\mathcal{K})$ such that $(\Gamma(\mathcal{K}), \circ)$ is a Leibniz algebra;
- (3) a bundle map $\kappa : \mathcal{K} \rightarrow TM \times \mathbb{R}$ which is a homomorphism into the Lie algebroid of first order differential operators satisfying the following properties,

$$(a). \langle Y \circ X, X \rangle = \langle Y, X \circ X \rangle, \quad (b). \kappa(X) \langle Y, Y \rangle = 2 \langle X \circ Y, Y \rangle.$$

Definition 2.4. A quasi-Jacobi bialgebroid is a triple $((A, \rho), \delta, \phi)$ consisting of a Jacobi algebroid (A, ρ) , a degree 1 derivation δ of the Schouten-Jacobi algebra $(\Gamma(\wedge^\bullet A), \llbracket \cdot, \cdot \rrbracket)$ and an element $\phi \in \Gamma(\wedge^3 A)$ such that $\delta^2 = \llbracket \phi, \cdot \rrbracket$ and $\delta\phi = 0$.

Definition 2.5. A quasi-Manin triple is a triple (\mathcal{K}, A, B) , where \mathcal{K} is a Courant-Jacobi algebroid, $A \subset \mathcal{K}$ is a Dirac structure and B is its transversal isotropic complement.

Remark 2.6. In [27], the notion of quasi-Jacobi bialgebroids has already been introduced, which is motivated by [30]. Our definition is motivated by [16]. One can easily recover the six conditions in their definition and some of the constructions are given in the proof of the next theorem.

Theorem 2.7. There is a one-to-one correspondence between quasi-Jacobi bialgebroids and quasi-Manin triples.

Proof. Let $((A, \rho), \delta, \phi)$ be a quasi-Jacobi bialgebroid. Define the bundle map $\rho_* : A^* \rightarrow TM \oplus \mathbb{R}$ by

$$\rho_*(\xi)(f) = \xi(\delta(f)), \quad \forall \xi \in \Gamma(A^*), f \in C^\infty(M). \quad (6)$$

Introduce a bracket $[\cdot, \cdot]_*$ on $\Gamma(A^*)$ by

$$[\xi, \eta]_*(X) = \rho_*(\xi)(\eta(X)) - \rho_*(\eta)(\xi(X)) - \delta(X)(\xi, \eta).$$

ρ_* is not a homomorphism but we have

$$\rho_*[\xi, \eta]_* = [\rho_*(\xi), \rho_*(\eta)] - \rho(\phi(\xi, \eta)).$$

Therefore, in general, $(A^*, [\cdot, \cdot]_*, \rho_*)$ is not a Jacobi algebroid. Let $\kappa : A \oplus A^* \rightarrow TM \oplus \mathbb{R}$ be the bundle map given by

$$\kappa(X + \xi) = \rho(X) + \rho_*(\xi).$$

Define a bracket $[\cdot, \cdot]$ on $\Gamma(A \oplus A^*)$ by

$$\begin{aligned} [X, Y] &= [X, Y], \quad \forall X, Y \in \Gamma(A), \\ [\xi, \eta] &= [\xi, \eta]_* + \phi(\xi, \eta, \cdot), \quad \forall \xi, \eta \in \Gamma(A^*), \\ [X, \xi] &= i_X \mathfrak{d}\xi - i_\xi \delta(X) + \mathfrak{d}(\xi(X)), \\ [\xi, X] &= -i_X \mathfrak{d}\xi + i_\xi \delta(X) + \delta(\xi(X)). \end{aligned}$$

Then $(A \oplus A^*, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \kappa)$ is a Courant-Jacobi algebroid such that A is a Dirac structure and A^* is its transversal isotropic complement.

Conversely, assume that $(\mathcal{K}, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \kappa)$ is a Courant-Jacobi algebroid and A is a Dirac structure with a transversal isotropic complement B , by using the pairing, we can identify B with A^* . Let $\rho = \kappa|_A$ be the restriction of κ on A , then (A, ρ) is a Jacobi algebroid. $\phi \in \Gamma(\wedge^3 A)$ is defined by

$$\phi(\xi, \eta, \gamma) = 2 \langle [\xi, \eta], \gamma \rangle, \quad \forall \xi, \eta, \gamma \in \Gamma(B). \quad (7)$$

Let $\rho_B = \kappa|_B$ be the restriction of κ to B and $[\cdot, \cdot]_B$ be the bracket on $\Gamma(B)$ such that

$$[\xi, \eta] - [\xi, \eta]_B \in \Gamma(A). \quad (8)$$

Define $\delta : \Gamma(\wedge^\bullet A) \rightarrow \Gamma(\wedge^{\bullet+1} A)$ by ρ_B and the bracket $[\cdot, \cdot]_B$ as (5), then $((A, \rho), \delta, \phi)$ is a quasi-Jacobi bialgebroid. ■

3 Jacobi quasi-Nijenhuis algebroids

A **Jacobi bi-vector field** on a Jacobi algebroid (A, ρ) is a bi-vector field $\pi \in \Gamma(\wedge^2 A)$ satisfying

$$[\pi, \pi] = 0.$$

Remark 3.1. It is called a Jacobi bi-vector field because in the case that $A = TM \times \mathbb{R}$ and the Lie algebroid structure on $TM \times \mathbb{R}$ is given by

$$[X + f, Y + g] = [X, Y] + Xg - Yf, \quad \forall X + f, Y + g \in \mathfrak{X}(M) \oplus C^\infty(M), \quad (9)$$

a bi-vector field is a pair (Λ, X) , where $\Lambda \in \mathfrak{X}^2(M)$ and $X \in \mathfrak{X}(M)$, and (Λ, X) is a Jacobi bi-vector field if and only if it is a Jacobi structure on M . See [17] and [4] for more details.

On $\Gamma(A^*)$, we can introduce a Lie bracket $[\cdot, \cdot]_\pi$, which is induced by a Jacobi bi-vector field π :

$$[\xi, \eta]_\pi = -\mathfrak{d}(\pi(\xi, \eta)) + \mathfrak{L}_{\pi^\sharp(\xi)}\eta - \mathfrak{L}_{\pi^\sharp(\eta)}\xi, \quad \forall \xi, \eta \in \Gamma(A^*). \quad (10)$$

Proposition 3.2. Let (A, ρ) be a Jacobi algebroid, $\pi \in \Gamma(\wedge^2 A)$ is a Jacobi bi-vector field if and only if $(A^*, \rho \circ \pi^\sharp)$ is a Jacobi algebroid, where the Lie algebroid structure on A^* is given by $(A^*, [\cdot, \cdot]_\pi, a \circ \pi^\sharp)$. In this case, we have $\mathfrak{d}_* \mathbf{1} = -\pi^\sharp(\mathfrak{d}\mathbf{1})$.

Proof. Since we also have the well known formula:

$$\pi^\sharp [\xi, \eta]_\pi - [\pi^\sharp(\xi), \pi^\sharp(\eta)] = \frac{1}{2} [\pi, \pi](\xi, \eta), \quad (11)$$

it follows that $[\cdot, \cdot]_\pi$ is a Lie bracket if and only if π is a Jacobi bi-vector field. In this case, it is obvious that $\rho \circ \pi^\sharp$ is a representation of the Lie algebroid $(A^*, [\cdot, \cdot]_\pi, a \circ \pi^\sharp)$. For any $\xi \in \Gamma(A)$, we have

$$\xi(\mathfrak{d}_* \mathbf{1}) = \rho \circ \pi^\sharp(\xi)(\mathbf{1}) = \mathfrak{d}\mathbf{1}(\pi^\sharp(\xi)) = -\xi(\pi^\sharp \mathfrak{d}\mathbf{1}),$$

which implies $\mathfrak{d}_* \mathbf{1} = -\pi^\sharp(\mathfrak{d}\mathbf{1})$ and the proof is finished. ■

Definition 3.3. Let (A, ρ) be a Jacobi algebroid, a Jacobi bi-vector field π and a $(1, 1)$ -tensor $N : A \rightarrow A$ are compatible if the following two conditions are satisfied:

$$N \circ \pi^\sharp = \pi^\sharp \circ N^* \quad \text{and} \quad C(\pi, N) = 0,$$

where

$$C(\pi, N)(\xi, \eta) \triangleq [\xi, \eta]_{\pi_N} - ([N^* \xi, \eta]_\pi + [\xi, N^* \eta]_\pi - N^* [\xi, \eta]_\pi), \quad \forall \xi, \eta \in \Gamma(A^*). \quad (12)$$

In the case where N is a Nijenhuis operator, i.e. $T(N) = 0$, the triple $((A, \rho), \pi, N)$ is said to be a **Jacobi-Nijenhuis algebroid**.

Remark 3.4. The notion of a Jacobi-Nijenhuis algebroid has already appeared in [4], where the author use the condition $\llbracket \pi, \pi_N \rrbracket = 0$ instead of $C(\pi, N) = 0$. In fact, if $C(\pi, N) = 0$, we can deduce that $\llbracket \pi, \pi_N \rrbracket = 0$, this is given by the next lemma

Lemma 3.5. Let (A, ρ) be a Jacobi algebroid, the Jacobi bi-vector field π is compatible with the $(1, 1)$ -tensor N , then we have

$$\llbracket \pi, \pi_N \rrbracket = 0.$$

Proof. By (11), we can obtain

$$\llbracket \pi, \pi_N \rrbracket(\xi, \eta) = \pi^\sharp \llbracket \xi, \eta \rrbracket_{\pi_N} + \pi^\sharp \circ N^* \llbracket \xi, \eta \rrbracket_\pi - [\pi^\sharp(\xi), \pi^\sharp(N^*\eta)] - [\pi^\sharp(N^*\xi), \pi^\sharp(\eta)].$$

If π and N are compatible, we have

$$\llbracket \xi, \eta \rrbracket_{\pi_N} = \llbracket N^*\xi, \eta \rrbracket_\pi + \llbracket \xi, N^*\eta \rrbracket_\pi - N^* \llbracket \xi, \eta \rrbracket_\pi.$$

The conclusion follows from the fact that $\pi^\sharp \llbracket \xi, \eta \rrbracket_\pi = [\pi^\sharp(\xi), \pi^\sharp(\eta)]$. ■

The degree 0 derivation i_N of $\Gamma(\wedge^\bullet A^*)$ is defined by

$$(i_N \omega)(X_1, \dots, X_k) = \sum_{i=1}^k \omega(X_1, \dots, NX_i, \dots, X_k), \quad \forall \omega \in \Gamma(\wedge^k A^*),$$

and we obtain a degree 1 differential operator $\mathfrak{d}_N : \Gamma(\wedge^\bullet A^*) \longrightarrow \Gamma(\wedge^{\bullet+1} A^*)$ by the following formula:

$$\mathfrak{d}_N = i_N \circ \mathfrak{d} - \mathfrak{d} \circ i_N.$$

Definition 3.6. A **Jacobi quasi-Nijenhuis algebroid** is a quadruple $((A, \rho), \pi, N, \phi)$, where (A, ρ) is a Jacobi algebroid, $\pi \in \Gamma(\wedge^2 A)$ is a Jacobi bi-vector field, $N \in \Gamma(A^* \otimes A)$ is compatible with π , and $\phi \in \Gamma(\wedge^3 A^*)$ satisfying $\mathfrak{d}\phi = 0$ and $\mathfrak{d}(i_N \phi) = 0$, such that

$$T(N)(X, Y) = \pi^\sharp(i_{X \wedge Y} \phi), \quad \forall X, Y \in \Gamma(A). \quad (13)$$

Theorem 3.7. The quadruple $((A, \rho), \pi, N, \phi)$ is a Jacobi quasi-Nijenhuis algebroid if and only if $((A^*, \rho \circ \pi^\sharp), \mathfrak{d}_N, \phi)$ is a quasi-Jacobi bialgebroid and $\mathfrak{d}\phi = 0$, where the Lie algebroid structure on A^* is given by $(A^*, \llbracket \cdot, \cdot \rrbracket_\pi, a \circ \pi^\sharp)$.

We need the following two lemmas to prove the theorem.

Lemma 3.8. Let (A, ρ) be a Jacobi algebroid. For a Jacobi bi-vector field π and a $(1, 1)$ -tensor $N : A \longrightarrow A$, the differential operator \mathfrak{d}_N is a derivation of the bracket $\llbracket \cdot, \cdot \rrbracket_\pi$ if and only if π and N are compatible.

Proof. This lemma is a generalization of Proposition 3.2 in [22], where one only need to prove that it holds for functions and 1-forms since it is a derivation with respect to the wedge product, \wedge . Here one can prove similarly that \mathfrak{d}_N is a derivation for functions and 1-forms, but since \mathfrak{d}_N is no longer a derivation with respect to the wedge product, \wedge , we can not say that it holds in general directly. But we will see that the obstruction of \mathfrak{d}_N to be a derivation is controlled by lower degree elements, therefore, we can still obtain that \mathfrak{d}_N is a derivation. In fact, since $\mathfrak{d}\omega = d\omega + \mathfrak{d}\mathbf{1} \wedge \omega$, for any $P \in \Gamma(\wedge^p A^*)$, we have

$$\mathfrak{d}_N P = \mathfrak{d}_N P + (i_N \mathfrak{d}\mathbf{1}) \wedge P. \quad (14)$$

Thus, for any $Q \in \Gamma(\wedge^q A^*)$, we have

$$\mathfrak{d}_N(P \wedge Q) = (\mathfrak{d}_N P) \wedge Q + (-1)^p P \wedge \mathfrak{d}_N Q - (i_N \mathfrak{d}\mathbf{1}) \wedge P \wedge Q.$$

On the other hand, by Proposition 3.2, for any $R \in \Gamma(\wedge^r A^*)$, we have

$$\llbracket P, Q \wedge R \rrbracket_\pi = \llbracket P, Q \rrbracket_\pi \wedge R + (-1)^{q(p+1)} Q \wedge \llbracket P, R \rrbracket_\pi - (i_{-\pi^\sharp(\mathfrak{d}\mathbf{1})} P) \wedge Q \wedge R.$$

Therefore, by direct computation, we have

$$\begin{aligned}
& \mathfrak{d}_N \llbracket P, Q \wedge R \rrbracket_\pi - \llbracket \mathfrak{d}_N P, Q \rrbracket_\pi \wedge R + (-1)^p \llbracket P, \mathfrak{d}_N(Q \wedge R) \rrbracket_\pi \\
= & (\mathfrak{d}_N \llbracket P, Q \rrbracket_\pi - \llbracket \mathfrak{d}_N P, Q \rrbracket_\pi + (-1)^p \llbracket P, \mathfrak{d}_N Q \rrbracket_\pi) \wedge R \\
& + (-1)^{p(q+1)} Q \wedge (\mathfrak{d}_N \llbracket P, R \rrbracket_\pi - \llbracket \mathfrak{d}_N P, R \rrbracket_\pi + (-1)^p \llbracket P, \mathfrak{d}_N R \rrbracket_\pi) \\
& - (\mathfrak{d}_N \llbracket P, \mathbf{1} \rrbracket_\pi - \llbracket \mathfrak{d}_N P, \mathbf{1} \rrbracket_\pi + (-1)^p \llbracket P, \mathfrak{d}_N \mathbf{1} \rrbracket_\pi) \wedge Q \wedge R.
\end{aligned}$$

This completes the proof. ■

Lemma 3.9. *Let (A, ρ) be a Jacobi algebroid. A Jacobi bi-vector field π and a $(1, 1)$ -tensor N are compatible. Then $\mathfrak{d}_N^2 = \llbracket \phi, \cdot \rrbracket_\pi$ is equivalent to (13) and $\pi^\sharp \circ (\mathfrak{d}\phi)_\flat = 0$, where $(\mathfrak{d}\phi)_\flat : \wedge^3 A \rightarrow A^*$ is the bundle map defined by $(\mathfrak{d}\phi)_\flat(X, Y, Z) = i_{X \wedge Y \wedge Z} \mathfrak{d}\phi$.*

Proof. By similar computations as in [31], we can easily obtain $\mathfrak{d}_N^2 - \llbracket \phi, \cdot \rrbracket_\pi$ vanishes on 0- and exact 1-forms if and only if $T(N)(X, Y) = \pi^\sharp(i_{X \wedge Y} \phi)$ and $\pi^\sharp \circ (\mathfrak{d}\phi)_\flat = 0$. But we should be very careful that \mathfrak{d}_N^2 and $\llbracket \phi, \cdot \rrbracket_\pi$ are no longer derivations with respect to the wedge product, \wedge , next we prove that we can still get $\mathfrak{d}_N^2 = \llbracket \phi, \cdot \rrbracket_\pi$. By (14), for any $P \in \Gamma(\wedge^p A^*)$, $Q \in \Gamma(\wedge^q A^*)$, we have

$$\mathfrak{d}_N^2(P \wedge Q) = (\mathfrak{d}_N^2 P) \wedge Q + P \wedge (\mathfrak{d}_N^2 Q) - (\mathfrak{d}_N i_N \mathfrak{d}\mathbf{1}) \wedge P \wedge Q.$$

On the other hand, we have

$$\llbracket \phi, P \wedge Q \rrbracket = \llbracket \phi, P \rrbracket \wedge Q + P \wedge \llbracket \phi, Q \rrbracket - (i_{-\pi^\sharp(\mathfrak{d}\mathbf{1})} \phi) \wedge P \wedge Q.$$

We only need to show

$$i_{\pi^\sharp(\mathfrak{d}\mathbf{1})} \phi = -\mathfrak{d}_N i_N \mathfrak{d}\mathbf{1}.$$

By direct computation, for any $X, Y \in \Gamma(A)$, we have

$$\begin{aligned}
i_{\pi^\sharp(\mathfrak{d}\mathbf{1})} \phi(X, Y) &= \phi(X, Y)(\pi^\sharp(\mathfrak{d}\mathbf{1})) = -\mathfrak{d}\mathbf{1}(\pi^\sharp(\phi(X, Y))), \\
\mathfrak{d}_N i_N \mathfrak{d}\mathbf{1}(X, Y) &= NX \mathfrak{d}\mathbf{1}(NY) - NY \mathfrak{d}\mathbf{1}(NX) - \mathfrak{d}\mathbf{1}(N[X, Y]_N) \\
&= \mathfrak{d}\mathbf{1}([NX, NY] - N[X, Y]_N) = \mathfrak{d}\mathbf{1}(\pi^\sharp(\phi(X, Y))).
\end{aligned}$$

This completes the proof. ■

The proof of Theorem 3.7: By Proposition 3.2, π is a Jacobi bi-vector field is equivalent to that $(A^*, \rho \circ \pi^\sharp)$ is a Jacobi algebroid. By Lemma 3.8, \mathfrak{d}_N is a derivation is equivalent to π and N are compatible. If $\mathfrak{d}(i_N \phi) = 0$ and $\mathfrak{d}\phi = 0$, we have $\mathfrak{d}_N \phi = i_N \mathfrak{d}\phi - \mathfrak{d}i_N \phi = 0$. Conversely, if $\mathfrak{d}_N \phi = \mathfrak{d}\phi = 0$, we have $\mathfrak{d}i_N \phi = 0$. By Lemma 3.9, the proof is finished. ■

Theorem 3.10. *Let $((A, \rho), \pi, N, \phi)$ be a Jacobi quasi-Nijenhuis algebroid, then we have*

$$\llbracket \pi_N, \pi_N \rrbracket(\xi, \eta) = -2\pi^\sharp(i_{\pi^\sharp(\xi) \wedge \pi^\sharp(\eta)} \phi).$$

Proof. By (11), for any $\xi, \eta \in \Gamma(A^*)$, we have

$$\begin{aligned}
\frac{1}{2} \llbracket \pi_N, \pi_N \rrbracket(\xi, \eta) &= N \circ \pi^\sharp(\llbracket N^* \xi, \eta \rrbracket_\pi + \llbracket \xi, N^* \eta \rrbracket - (\mathfrak{L}_{\pi^\sharp(\xi)} N^* \eta - \mathfrak{L}_{\pi^\sharp(\eta)} N^* \xi - \mathfrak{d}\pi(N^* \xi, \eta))) \\
&\quad - \pi^\sharp[N^* \xi, N^* \eta]_\pi \\
&= N \circ \pi^\sharp(\llbracket N^* \xi, \eta \rrbracket_\pi + \llbracket \xi, N^* \eta \rrbracket - N^* \llbracket \xi, \eta \rrbracket_\pi) - \pi^\sharp[N^* \xi, N^* \eta]_\pi \\
&= -T(N)(\pi^\sharp(\xi), \pi^\sharp(\eta)) \\
&= -\pi^\sharp(i_{\pi^\sharp(\xi) \wedge \pi^\sharp(\eta)} \phi).
\end{aligned}$$

The second equality holds is because $C(\pi, N) = 0$. Since π is a Jacobi bi-vector field, we get the third equality. The last equality follows from the definition of a Jacobi quasi-Nijenhuis algebroid. ■

4 Generalized complex structures

Let (A, ρ) be a Jacobi algebroid. There is a natural pairing $\langle \cdot, \cdot \rangle$ on $A \oplus A^*$ which is given by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)), \quad \forall X, Y \in \Gamma(A), \xi, \eta \in \Gamma(A^*). \quad (15)$$

and we can introduce a bracket on the section space $\Gamma(A) \oplus \Gamma(A^*)$ which is given by

$$[X + \xi, Y + \eta] = [X, Y] + \mathfrak{L}_X \eta - \mathfrak{L}_Y \xi + \mathfrak{d}(\xi(Y)). \quad (16)$$

Obviously, $(A \oplus A^*, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ is a Courant-Jacobi algebroid, where $\rho(X + \xi) = \rho(X)$. In this section we study generalized complex structures on this Courant-Jacobi algebroid and we will see that they are related with Jacobi quasi-Nijenhuis algebroids in the same way as how generalized complex structures on a manifold are related with Poisson quasi-Nijenhuis structures. In the following two sections, we will see that generalized complex structures on this Courant-Jacobi algebroid unify the usual generalized complex structures on an even-dimensional manifold and generalized contact structures on an odd-dimensional manifold.

Definition 4.1. *A generalized complex structure on the Courant-Jacobi algebroid $(A \oplus A^*, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ is a bundle map $\mathcal{J} : A \oplus A^* \rightarrow A \oplus A^*$ satisfying the algebraic properties*

$$\mathcal{J}^2 = -\text{Id}, \quad \langle \mathcal{J}u, \mathcal{J}v \rangle = \langle u, v \rangle, \quad \forall u, v \in \Gamma(A) \oplus \Gamma(A^*) \quad (17)$$

and the integrability condition

$$[\mathcal{J}u, \mathcal{J}v] - [u, v] - \mathcal{J}([\mathcal{J}u, v] + [u, \mathcal{J}v]) = 0, \quad (18)$$

where $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$ are given by (15) and (16) respectively.

By (17), \mathcal{J} must be of the form

$$\mathcal{J} = \begin{pmatrix} N & \pi^\sharp \\ \sigma_b & -N^* \end{pmatrix}, \quad (19)$$

where $\pi \in \Gamma(\wedge^2 A)$, $\sigma \in \Gamma(\wedge^2 A^*)$, $N \in \Gamma(A^* \otimes A)$, in which the following conditions are satisfied:

$$N \circ \pi^\sharp = \pi^\sharp \circ N^*, \quad N^2 + \pi^\sharp \circ \sigma_b = -\text{Id}, \quad N^* \circ \sigma_b = \sigma_b \circ N.$$

Similar as the proof of Proposition 2.2 in [9], we have

Proposition 4.2. *For any generalized complex structure \mathcal{J} given by (19) on the Courant-Jacobi algebroid $(A \oplus A^*, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$, π is a Jacobi bivector field. Thus there is an induced Jacobi structure on the base manifold M .*

Remark 4.3. *The author gives his warmest thanks to the referee for pointing out this fact.*

We deform a Courant-Jacobi algebroid using a bundle map \mathcal{J} . More precisely, we introduce a new inner product $\langle \cdot, \cdot \rangle_{\mathcal{J}}$, a new bracket $[\cdot, \cdot]_{\mathcal{J}}$ and a new anchor $\rho_{\mathcal{J}}$ by

$$\begin{aligned} \langle u, v \rangle_{\mathcal{J}} &= \langle \mathcal{J}u, \mathcal{J}v \rangle, \\ [u, v]_{\mathcal{J}} &= [\mathcal{J}u, v] + [u, \mathcal{J}v] - \mathcal{J}[u, v], \\ \rho_{\mathcal{J}} &= \rho \circ \mathcal{J}. \end{aligned}$$

Proposition 4.4. *Let $\mathcal{J} : A \oplus A^* \rightarrow A \oplus A^*$ be a bundle map given by (19), then \mathcal{J} is a generalized complex structure if and only if $(A \oplus A^*, \langle \cdot, \cdot \rangle_{\mathcal{J}}, [\cdot, \cdot]_{\mathcal{J}}, \rho_{\mathcal{J}})$ is a Courant-Jacobi algebroid such that \mathcal{J} is a Courant-Jacobi algebroid morphism from $(A \oplus A^*, \langle \cdot, \cdot \rangle_{\mathcal{J}}, [\cdot, \cdot]_{\mathcal{J}}, \rho_{\mathcal{J}})$ to $(A \oplus A^*, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$.*

Proof. If \mathcal{J} given by (19) is a generalized complex structure, first we note that $\langle \cdot, \cdot \rangle_{\mathcal{J}} = \langle \cdot, \cdot \rangle$. $[\cdot, \cdot]_{\mathcal{J}}$ is still a Leibniz bracket follows from (18). Also by (18), for any $u, v \in \Gamma(A \oplus A^*)$, we have

$$\rho_{\mathcal{J}}([u, v]_{\mathcal{J}}) = \rho \circ \mathcal{J} [u, v]_{\mathcal{J}} = \rho [\mathcal{J}u, \mathcal{J}v] = [\rho \circ \mathcal{J}u, \rho \circ \mathcal{J}v] = [\rho_{\mathcal{J}}u, \rho_{\mathcal{J}}v],$$

which implies $\rho_{\mathcal{J}}$ is a homomorphism. Next we verify that the conditions (a), (b) in Definition 2.3 are satisfied. Since \mathcal{J} preserves the inner product $\langle \cdot, \cdot \rangle$, we have

$$\begin{aligned} \langle [u, v]_{\mathcal{J}}, v \rangle &= \langle \mathcal{J} [u, v]_{\mathcal{J}}, \mathcal{J}v \rangle = \langle [\mathcal{J}u, \mathcal{J}v], \mathcal{J}v \rangle = \langle \mathcal{J}u, [\mathcal{J}v, \mathcal{J}v] \rangle = \langle \mathcal{J}u, \mathcal{J} [v, v]_{\mathcal{J}} \rangle \\ &= \langle u, [v, v]_{\mathcal{J}} \rangle, \end{aligned}$$

which implies that Condition (a) in Definition 2.3 is satisfied. Similarly, we have

$$\rho_{\mathcal{J}}(u) \langle v, v \rangle = \rho(\mathcal{J}u) \langle \mathcal{J}v, \mathcal{J}v \rangle = 2 \langle [\mathcal{J}u, \mathcal{J}v], \mathcal{J}v \rangle = 2 \langle \mathcal{J} [u, v]_{\mathcal{J}}, \mathcal{J}v \rangle = 2 \langle [u, v]_{\mathcal{J}}, v \rangle,$$

which implies that Condition (b) is satisfied. Thus $(A \oplus A^*, \langle \cdot, \cdot \rangle_{\mathcal{J}}, [\cdot, \cdot]_{\mathcal{J}}, \rho_{\mathcal{J}})$ is a Courant-Jacobi algebroid. Furthermore, \mathcal{J} is a Courant-Jacobi algebroid morphism from Courant-Jacobi algebroid $(A \oplus A^*, \langle \cdot, \cdot \rangle_{\mathcal{J}}, [\cdot, \cdot]_{\mathcal{J}}, \rho_{\mathcal{J}})$ to $(A \oplus A^*, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ is obvious. The converse part is straightforward and the proof is completed. ■

Theorem 4.5. *Let $\mathcal{J} : A \oplus A^* \rightarrow A \oplus A^*$ be a bundle map given by (19). Then $(A \oplus A^*, \langle \cdot, \cdot \rangle_{\mathcal{J}}, [\cdot, \cdot]_{\mathcal{J}}, \rho_{\mathcal{J}})$ is a Courant-Jacobi algebroid if and only if $((A, \rho), \pi, N, \mathfrak{d}\sigma)$ is a Jacobi quasi-Nijenhuis algebroid.*

Proof. One can easily see that for all $X, Y \in \Gamma(A)$ and $\xi, \eta \in \Gamma(A^*)$, we have

$$\begin{aligned} [X, Y]_{\mathcal{J}} &= [X, Y]_N + \mathfrak{d}\sigma(X, Y, \cdot), \\ [\xi, \eta]_{\mathcal{J}} &= \llbracket \xi, \eta \rrbracket_{\pi}, \\ [X, \xi]_{\mathcal{J}} &= [X, \pi^{\sharp}(\xi)] - \pi^{\sharp} \mathfrak{L}_X \xi + \mathfrak{L}_{NX} \xi - \mathfrak{L}_X(N^* \xi) + N^* \mathfrak{L}_X \xi, \\ [\xi, X]_{\mathcal{J}} &= -[X, \xi]_{\mathcal{J}} - \mathcal{J} \mathfrak{d}(\xi(X)). \end{aligned}$$

Therefore, if $(A \oplus A^*, \langle \cdot, \cdot \rangle_{\mathcal{J}}, [\cdot, \cdot]_{\mathcal{J}}, \rho_{\mathcal{J}})$ is a Courant-Jacobi algebroid, A^* is a Dirac structure, and A is its isotropic transversal complement. By Theorem 2.7, we obtain a quasi-Jacobi bialgebroid. More precisely, we have

$$\rho_A = \rho \circ N, \quad [\cdot, \cdot]_A = [\cdot, \cdot]_N, \quad \delta = \mathfrak{d}_N, \quad \phi = \mathfrak{d}\sigma,$$

and the quasi-Jacobi bialgebroid is given by $((A^*, \rho \circ \pi^{\sharp}), \mathfrak{d}_N, \mathfrak{d}\sigma)$, or equivalently $((A, \rho), \pi, N, \mathfrak{d}\sigma)$ is a Jacobi quasi-Nijenhuis algebroid.

Conversely, assume $((A, \rho), \pi, N, \mathfrak{d}\sigma)$ is a Jacobi quasi-Nijenhuis algebroid, then $((A^*, \rho \circ \pi^{\sharp}), \mathfrak{d}_N, \mathfrak{d}\sigma)$ is a quasi-Jacobi bialgebroid and its double is a Courant-Jacobi algebroid, denote by E . It is straightforward to see that E is isomorphic to $(A \oplus A^*, \langle \cdot, \cdot \rangle_{\mathcal{J}}, [\cdot, \cdot]_{\mathcal{J}}, \rho_{\mathcal{J}})$. ■

By Proposition 4.4 and Theorem 4.5, we have

Theorem 4.6. *Let (A, ρ) be a Jacobi algebroid. Assume that $\mathcal{J} : A \oplus A^* \rightarrow A \oplus A^*$ is a bundle map given by (19), then \mathcal{J} is a generalized complex structure is equivalent to that $((A, \rho), \pi, N, \mathfrak{d}\sigma)$ is a Jacobi quasi-Nijenhuis algebroid such that \mathcal{J} is a Courant-Jacobi algebroid morphism from Courant-Jacobi algebroid $(A \oplus A^*, \langle \cdot, \cdot \rangle_{\mathcal{J}}, [\cdot, \cdot]_{\mathcal{J}}, \rho_{\mathcal{J}})$ to $(A \oplus A^*, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$, where the first one corresponds to the quasi-Jacobi bialgebroid $((A^*, \rho \circ \pi^{\sharp}), \mathfrak{d}_N, \mathfrak{d}\sigma)$.*

5 Generalized complex structures on $\mathcal{T}M$

In this section, we consider the case where the vector bundle A is the tangent bundle TM of a manifold M . Since the tangent Lie algebroid is a special Jacobi algebroid, it follows that generalized complex structures on a manifold M is a special case of what we discussed in the last section. Next we first

recall the notion of generalized complex structures on a manifold M and then we deform the tangent Lie algebroid to be a Jacobi algebroid and study its generalized complex structures. Consider the generalized tangent bundle

$$\mathcal{T}M := TM \oplus T^*M,$$

on its section space $\Gamma(\mathcal{T}M)$, there is a well known Dorfman bracket, explicitly,

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - L_Y\xi + d(\xi(Y)), \quad \forall X + \xi, Y + \eta \in \Gamma(\mathcal{T}). \quad (20)$$

Definition 5.1. *A generalized complex structure on a manifold M is a bundle map $\mathcal{J} : \mathcal{T}M \longrightarrow \mathcal{T}M$ satisfying the algebraic properties:*

$$\mathcal{J}^2 = -\text{Id} \quad \text{and} \quad \langle \mathcal{J}(u), \mathcal{J}(v) \rangle = \langle u, v \rangle$$

and the integrability condition:

$$[\mathcal{J}(u), \mathcal{J}(v)] - [u, v] - \mathcal{J}([\mathcal{J}(u), v] + [u, \mathcal{J}(v)]) = 0, \quad \forall u, v \in \Gamma(\mathcal{T}).$$

We consider the bracket (20) deformed by a 1-cocycle ϕ_0 in the deRham cohomology. More precisely, the new bracket $[\cdot, \cdot]$ is given by

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - L_Y\xi + d(\xi(Y)) + (i_X\phi_0)\eta - i_Y(\phi_0 \wedge \xi). \quad (21)$$

It is easy to see that $(\Gamma(\mathcal{T}M), [\cdot, \cdot])$ is still a Leibniz algebra, but it is not a Courant algebroid since

$$[X + \xi, Y + \eta] = d(\xi(Y)) + \xi(Y)\phi_0.$$

In fact, ϕ_0 decides a representation $\rho : TM \longrightarrow TM \oplus \mathbb{R}$ which is given by

$$\rho(X) = X + \phi_0(X). \quad (22)$$

Now (TM, ρ) is a Jacobi algebroid. We rewrite (21) as

$$[X + \xi, Y + \eta] = [X, Y] + \mathfrak{L}_X\eta - \mathfrak{L}_Y\xi + d(\xi(Y)). \quad (23)$$

Therefore, we obtain a Courant-Jacobi algebroid $(\mathcal{T}M, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$, where $\langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho$ are given by (15), (23) and (22) respectively.

Proposition 5.2. *With the above notations, consider generalized complex structures of the Courant-Jacobi algebroid $(\mathcal{T}M, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$, we have*

- (1). *For any $N : TM \longrightarrow TM$ which is a Nijenhuis operator and satisfies $N^2 = -\text{Id}$, $\begin{pmatrix} N & 0 \\ 0 & -N^* \end{pmatrix}$ is a generalized complex structure.*
- (2). *For any $\omega \in \Omega^2(M)$, $\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$ is a generalized complex structure if and only if $d\omega = 0$.*
- (3). *For a $(1,1)$ -tensor N satisfying $N^2 = -\text{Id}$ and $\pi \in \mathfrak{X}^2(M)$, $\begin{pmatrix} N & \pi \\ 0 & -N^* \end{pmatrix}$ is a generalized complex structure if and only if*

$$\begin{aligned} N \circ \pi^\sharp &= \pi^\sharp \circ N^*, \\ [\pi^\sharp(\xi), \pi^\sharp(\eta)] &= \pi^\sharp[\xi, \eta]_\pi, \\ N^*(\llbracket \xi, \eta \rrbracket_\pi) &= \mathfrak{L}_{\pi^\sharp(\xi)}(N^*\eta) - \mathfrak{L}_{\pi^\sharp(\eta)}(N^*\xi) - d\pi(N^*\xi, \eta), \end{aligned}$$

where $\llbracket \xi, \eta \rrbracket_\pi$ is given by (10).

Corollary 5.3. *If we write (2) and (3) in the above proposition in term of ϕ_0 , we have*

- (1). *For any nondegenerate conformal symplectic structure (ϕ_0, ω) , i.e. $\omega \in \Omega^2(M)$ is nondegenerate and satisfies $d\omega = \phi_0 \wedge \omega$, $\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$ is a generalized complex structure.*
- (2). *For a $(1,1)$ -tensor N satisfying $N^2 = -\text{Id}$ and $\pi \in \mathfrak{X}^2(M)$ satisfying*

$$\begin{aligned} N \circ \pi^\sharp &= \pi^\sharp \circ N^*, \\ [\pi^\sharp(\xi), \pi^\sharp(\eta)] &= \pi^\sharp[\xi, \eta]_\pi + \frac{1}{2}i_{\phi_0}(\pi \wedge \pi)(\xi, \eta) = 0, \\ N^*([\xi, \eta]_\pi + \pi(\eta, \xi)\phi_0) &= L_{\pi^\sharp(\xi)}(N^*\eta) - L_{\pi^\sharp(\eta)}(N^*\xi) - d\pi(N^*\xi, \eta) + \pi(\eta, N^*\xi)\phi_0, \\ \begin{pmatrix} N & \pi \\ 0 & -N^* \end{pmatrix} &\text{ is a generalized complex structure, where } [\xi, \eta]_\pi \text{ is given by} \end{pmatrix}$$

$$[\xi, \eta]_\pi = L_{\pi^\sharp(\xi)}\eta - L_{\pi^\sharp(\eta)}\xi - d\pi(\xi, \eta).$$

Remark 5.4. *By (1) in Proposition 5.2, we can see that there are some generalized complex structures which are stable when the bracket is deformed by (21). By (2), we see that how a conformal symplectic structure on a manifold relates with a generalized complex structure.*

6 Generalized complex structures on $\mathcal{E}^1(M)$

Note that only even-dimensional manifolds can have generalized complex structures. In [19], the authors give the odd-dimensional analogue of the concept of generalized complex structures. Denote $(TM \oplus \mathbb{R}) \oplus (T^*M \oplus \mathbb{R})$ by $\mathcal{E}^1(M)$, and there is a natural bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{E}^1(M)$ defined by:

$$\langle (X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2) \rangle = \frac{1}{2}(\alpha_2(X_1) + \alpha_1(X_2) + f_1g_2 + f_2g_1). \quad (24)$$

There is also a bracket which is given by

$$\begin{aligned} &[(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)] \\ &= ([X_1, X_2], X_1f_2 - X_2f_1) + \tilde{\mathfrak{L}}_{(X_1, f_1)}(\alpha_2, g_2) - i_{(X_2, f_2)}\tilde{d}(\alpha_1, g_1). \end{aligned} \quad (25)$$

For more information about $\tilde{\mathfrak{L}}$ and \tilde{d} , see [19].

Definition 6.1. *A generalized contact structure on a $(2n+1)$ -dimensional manifold M is a bundle map $\mathcal{J} : \mathcal{E}^1(M) \longrightarrow \mathcal{E}^1(M)$ satisfying the algebraic properties:*

$$\mathcal{J}^2 = -\text{Id} \quad \text{and} \quad \langle \mathcal{J}(u), \mathcal{J}(v) \rangle = \langle u, v \rangle$$

and the integrability condition:

$$[\mathcal{J}(u), \mathcal{J}(v)] - [u, v] - \mathcal{J}([\mathcal{J}(u), v] + [u, \mathcal{J}(v)]) = 0, \quad \forall u, v \in \Gamma(\mathcal{E}^1(M)).$$

Here, $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$ are given by (24) and (25) respectively.

We know that $TM \oplus \mathbb{R} = \mathfrak{D}(M \times \mathbb{R})$, the covariant differential operator bundle of the trivial line bundle $M \times \mathbb{R}$. In fact, we also have $T^*M \oplus \mathbb{R} = \mathfrak{J}(M \times \mathbb{R})$, the first jet bundle of the trivial line bundle $M \times \mathbb{R}$. In [6], the authors proved that for any vector bundle E , the first jet bundle $\mathfrak{J}E$ may be considered as an E -dual bundle of $\mathfrak{D}E$, i.e.

$$\mathfrak{J}E \cong \{ \nu \in \text{Hom}(\mathfrak{D}E, E) \mid \nu(\Phi) = \Phi \circ \nu(\text{Id}_E), \quad \forall \Phi \in \text{gl}(E) \} \subset \text{Hom}(\mathfrak{D}E, E).$$

We can introduce an E -valued pairing $(\cdot, \cdot)_E$ on $\mathfrak{D}E \oplus \mathfrak{J}E$ by

$$(\mathfrak{d} + \mu, \mathfrak{t} + \nu)_E = \frac{1}{2}(\mu(\mathfrak{t}) + \nu(\mathfrak{d})) = \frac{1}{2}(\langle \mathfrak{t}, \mu \rangle_E + \langle \mathfrak{d}, \nu \rangle_E), \quad \forall \mathfrak{d} + \mu, \mathfrak{t} + \nu \in \mathfrak{D}E \oplus \mathfrak{J}E. \quad (26)$$

Furthermore, for any $\mathfrak{d} \in \Gamma(\mathfrak{D}E)$, the Lie derivative $\mathfrak{L}_{\mathfrak{d}} : \Gamma(\mathfrak{J}E) \rightarrow \Gamma(\mathfrak{J}E)$ is defined by:

$$\langle \mathfrak{L}_{\mathfrak{d}}\mu, \mathfrak{d}' \rangle_E \triangleq \mathfrak{d} \langle \mu, \mathfrak{d}' \rangle_E - \langle \mu, [\mathfrak{d}, \mathfrak{d}']_{\mathfrak{D}} \rangle_E, \quad \forall \mu \in \Gamma(\mathfrak{J}E), \mathfrak{d}' \in \Gamma(\mathfrak{D}E).$$

On the section space $\Gamma(\mathfrak{D}E \oplus \mathfrak{J}E)$, we can define a bracket as follows

$$[\mathfrak{d} + \mu, \mathfrak{t} + \nu] \triangleq [\mathfrak{d}, \mathfrak{t}]_{\mathfrak{D}} + \mathfrak{L}_{\mathfrak{d}}\nu - \mathfrak{L}_{\mathfrak{t}}\mu + \mathfrak{d}\mu(\mathfrak{t}). \quad (27)$$

Therefore, we have $\mathcal{E}^1(M) = \mathfrak{D}(M \times \mathbb{R}) \oplus \mathfrak{J}(M \times \mathbb{R})$, and we can rewrite (25) by (27) and (24) by

$$\langle \mathfrak{d} + \mu, \mathfrak{t} + \nu \rangle = \frac{1}{2}(\mu(\mathfrak{t}) + \nu(\mathfrak{d})), \quad \forall \mathfrak{d} + \mu, \mathfrak{t} + \nu \in \mathfrak{D}(M \times \mathbb{R}) \oplus \mathfrak{J}(M \times \mathbb{R}). \quad (28)$$

The following proposition is straightforward.

Proposition 6.2. *The quadruple $(\mathcal{E}^1(M), \langle \cdot, \cdot \rangle, [\cdot, \cdot], \text{Id})$ is a Courant-Jacobi algebroid, where $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$ are given by (28) and (27) and $\text{Id}(\mathfrak{d} + \mu) = \mathfrak{d}$. Therefore, generalized contact structures on an odd dimensional manifold is exactly generalized complex structures on this Courant-Jacobi algebroid.*

Example 6.3. We consider generalized complex structures \mathcal{J} of the type $\begin{pmatrix} N & 0 \\ 0 & -N^* \end{pmatrix}$, where $N : TM \oplus \mathbb{R} \rightarrow TM \oplus \mathbb{R}$ is a bundle map. Then the requirements are $N^2 = -\text{Id}$ and $T(N) = 0$ which are similar as the condition of a usual generalized complex structure. More simply, if we consider $N = \begin{pmatrix} \varphi & -Y \\ \eta & 0 \end{pmatrix}$, where $\varphi \in \Gamma(T^*M \otimes TM)$, $Y \in \mathfrak{X}(M)$ is a vector field and $\eta \in \Omega^1(M)$ is a 1-form, then the condition $N^2 = -\text{Id}$ is equivalent to

$$\begin{pmatrix} \varphi^2 - \eta \otimes Y & -\varphi(Y) \\ \eta \circ \varphi & -\eta(Y) \end{pmatrix} = -\text{Id}.$$

Therefore,

$$\eta(Y) = 1, \quad \varphi^2 - \eta \otimes Y = -\text{Id}, \quad (29)$$

$$\varphi(Y) = 0, \quad \eta \circ \varphi = 0. \quad (30)$$

But, we should note that (30) follows from (29). In fact, if $\eta(Y) = 1$ and

$$\varphi^2(X) = -X + \eta(X)Y, \quad \forall X \in \mathfrak{X}(M), \quad (31)$$

first we have $\varphi^2(Y) = 0$. In (31), substitute X by $\varphi(Y)$, we obtain $\varphi(Y) = \eta(\varphi(Y))Y$. Acting by φ , we obtain

$$0 = \varphi^2(Y) = \varphi(\eta(\varphi(Y))Y) = \eta(\varphi(Y))\varphi(Y) = \eta(\varphi(Y))^2Y,$$

which implies $\eta(\varphi(Y)) = 0$, and therefore $\varphi(Y) = 0$. Thus, (φ, Y, η) is an **almost contact structure**. Furthermore, by straightforward computations, $T(N) = 0$ is equivalent to

$$T(\varphi)(X_1, X_2) + \mathfrak{d}\eta(X_1, X_2)Y = 0, \quad \forall X_1, X_2 \in \mathfrak{X}(M),$$

which is equivalent to the condition that (φ, Y, η) is a **normal contact structure**, where $T(\varphi)$ is the Nijenhuis torsion of φ , see (1).

Example 6.4. We consider generalized complex structures \mathcal{J} of the type $\begin{pmatrix} 0 & \Upsilon \\ \Theta & 0 \end{pmatrix}$, where $\Theta : TM \oplus \mathbb{R} \rightarrow T^*M \oplus \mathbb{R}$ and $\Upsilon : T^*M \oplus \mathbb{R} \rightarrow TM \oplus \mathbb{R}$ are bundle maps. Evidently, $\mathcal{J}^2 = -\text{Id}$ implies that $\Upsilon = -\Theta^{-1}$. $\mathcal{J}^* = -\mathcal{J}$ implies Θ is skew-symmetric. At last, from the integrability condition, we obtain that $d(\Theta) = 0$. Since Θ is skew-symmetric, we can assume $\Theta = \begin{pmatrix} \omega & \eta \\ -\eta & 0 \end{pmatrix}$, where $\omega \in \Omega^2(M)$ is a 2-form and $\eta \in \Omega^1(M)$ is a 1-form such that $\eta \wedge \omega^n \neq 0$ to insure that Θ is invertible.

If we let $\frac{\partial}{\partial t}$ as a basis of $\Gamma(M \times \mathbb{R})$ in $\Gamma(TM \oplus \mathbb{R})$, then any $\mathfrak{d} \in \Gamma(TM \oplus \mathbb{R})$ can be write as $\mathfrak{d} = X + f \frac{\partial}{\partial t}$ for some $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Dually, any $\mu \in \Gamma(T^*M \oplus \mathbb{R})$ can be write as $\mu = \xi + g dt$. Then it is easy to see $\Theta = \begin{pmatrix} \omega & \eta \\ -\eta & 0 \end{pmatrix} \in \wedge^2 \Gamma(T^*M \oplus \mathbb{R})$ is given by $\omega + dt \wedge \eta$. Since the representation of the Jacobi algebroid $TM \oplus \mathbb{R}$ is the identity map, we have $d\mathbf{1} = dt$. Thus we have

$$d\Theta = d(\omega + dt \wedge \eta) = d\omega + dt \wedge (\omega - d\eta).$$

So $d\Theta = 0$ precisely means that $\omega - d\eta = 0$, i.e. $\omega = d\eta$. Since we also have $\eta \wedge \omega^n \neq 0$, it follows that η is a **contact structure**.

References

- [1] P. Antunes, Poisson quasi-Nijenhuis structures with background, *Lett. Math. Phys.* (2008) 86:33-45.
- [2] J.F. Carinena, J. Grabowski and G. Marmo, Contractions: Nijenhuis and Saletan tensors for general algebraic structures, *J. Phys. A: Math. Gen.* 34 (2001), 3769-3789.
- [3] J.F. Carinena, J. Grabowski and G. Marmo, Courant algebroid and Lie bialgebroid contractions, *J. Phys. A: Math. Gen.* 37:5189-5202, 2004.
- [4] R. Caseiro and J. M. Nunes da Costa, Jacobi-Nijenhuis algebroids and their modular classes, *J. Phys. A: Math. Gen.*, Volume 40, Issue 44, pp. 13311-13330 (2007).
- [5] B. Chen and Y.-H. Sheng, Poisson-Nijenhuis structures on oriented 3D-manifolds, *Rep. Math. Phys.*, 61 (2008), No. 3, 361-380.
- [6] Z. Chen and Z.-J. Liu, Omni-Lie algebroids, arXiv:0710.1923.
- [7] Z. Chen, Z.-J. Liu and Y.-H. Sheng, Dirac structures of omni-Lie algebroids, arXiv:0802.3819.
- [8] Z. Chen and Z.-J. Liu and Y.-H. Sheng, *E*-Courant algebroids, arXiv:0805.4093.
- [9] M. Crainic, Generalized complex structures and Lie brackets, arXiv:math.DG/0412097.
- [10] P. Dazord, A. Lichnerowicz and C-M. Marle, Structures locales des variétés de Jacobi, *J. Math. Pures Appl.* **70**(1991), 101-152.
- [11] J. Grabowski, Courant-Nijenhuis tensors and generalized geometries, *Monografías de la Real Academia de Ciencias de Zaragoza* 29 (2006), 101-112.
- [12] J. Grabowski and G. Marmo, Jacobi structures revisited, *J. Phys. A: Math. Gen.* 34 (2001), 10975-90.
- [13] J. Grabowski and G. Marmo, The graded Jacobi algebras and (co)homology, *J. Phys. A: Math. Gen.* 36 (2003), 161-81.
- [14] M. Gualtieri, *Generalized complex geometry*. Preprint math.DG/0404451.
- [15] N. Hitchin, Generalized Calabi-Yau manifolds, *Quart. J. Math.* 54 (3) (2003), 281-308.

- [16] D. Iglesias, C. Laurent-Gengoux and P. Xu, Universal lifting theorem and quasi-Poisson groupoids, arXiv:math/0507396.
- [17] D. Iglesias, B. López, J.C. Marrero and E. Padrón, Triangular generalized Lie bialgebroids: Homology and cohomology theories, *Proceedings of Workshop Lie algebroids and related topics in differential geometry (Varsovia, 2000)*, *Banach Center Publications* 54 (2001), 111-133.
- [18] D. Iglesias and J.C. Marrero, Generalized Lie bialgebroids and Jacobi structures, *J. Geom. Phys.* 40, (2001) 176-200.
- [19] D. Iglesias-Ponte and A. Wade, Contact manifolds and generalized complex structures, *J. Geom. Phys.* 53, (2005) 249-258.
- [20] A. Kirillov, Local Lie algebras, *Russian Math. Surveys*, **31**(1976), 55-76.
- [21] Y. Kosmann-Schwarzbach and F. Magri, Poisson-Nijenhuis structures, *Annales de l'institut Henri Poincaré (A) Physique théorique*, **53** no. 1 (1990), 35-81.
- [22] Y. Kosmann-Schwarzbach, The Lie bialgebroid of a Poisson-Nijenhuis manifold, *Lett. Math. Phys.* 38 (4): 421-428, 1996.
- [23] Z.-J. Liu, A. Weinstein and P. Xu, Manin triples for Lie bialgebroids, *J. Diff. Geom.*, **45**(1997), 547-574.
- [24] K. Mackenzie and P. Xu, Lie bialgebroids and Poisson groupoids, *Duke Math. J.* **73** (2) (1994), 415-452.
- [25] K. Mackenzie, *General theories of Lie groupoids and Lie algebroids*, Cambridge University Press, 2005.
- [26] J. Nunes da Costa and F. Petalidou, On quasi-Jacobi and Jacobi-quasi bialgebroids, *Lett. Math. Phys.* 80 (2007), No. 2, 155-169.
- [27] J. Nunes da Costa and F. Petalidou, Twisted Jacobi manifolds, twisted Dirac-Jacobi structures and quasi-Jacobi bialgebroids, *J. Phys. A: Math. Gen.* 39, 10449-10475.
- [28] J. M. Nunes da Costa and J. Clemente-Gallardo, Dirac Structures for generalized Lie bialgebroids, *J. Phys. A: Math. Gen.* 37 (2004) 2671-2692.
- [29] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, PhD thesis, UC Berkeley, 1999, arXiv:math.DG/9910078.
- [30] D. Roytenberg, Quasi-Lie bialgebroids and twisted Poisson manifolds, *Lett. Math. Phys.* 61 (2002), No. 2, 123-137.
- [31] M. Stiénon and P. Xu, Poisson quasi-Nijenhuis manifolds, *Comm. Math. Phys.* 270 (2007), No. 3, 709-725.
- [32] A. Wade, Generalizations of Poisson-Nijenhuis structures, *J. Geom. Phys.* 39 (2001), No. 3, 217-232.